A Lightweight Encoder and Decoder for Non-Binary Polar Codes

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***Abstract*—In this paper, we associate the Cyclic Code-Shift Keying (CCSK) modulation to Non-Binary Polar (NB-Polar) codes to have good decoding performance at ultra-low signal-to-noise ratios (SNRs). We show that the kernel transformation of order 2 using kernel coefficients equal to 1 reduces the complexity of the encoder and decoder and does not degrade error-correcting performance. We consider the Successive-Cancellation Min-Sum (SC-MS) decoder. To greatly reduce the complexity of the SC-MS decoder, we choose a small number of most meaningful values at the check node processing. Our simulation results show that the optimized SC-MS decoder presents a negligible performance degradation with respect to the SC decoder.**

***Index Terms*—Error correction, Non-Binary Polar Codes, Successive-Cancellation (SC) Min-Sum decoders, Channel Polarization, Low-complexity encoder and decoder.**

I. Introduction

It is well known that Binary Polar (B-Polar) codes [1] are provable capacity-achieving error correction codes for binary-input discrete memoryless channels. Non-Binary Polar (NB-Polar) codes [2]–[7], derived from B-Polar codes, are defined over the Galois Field (GF) $\mathbb{F}_q$, $q > 2$. NB-Polar codes, compared to B-Polar codes, reduce the probability of frame error by processing multiple bits in parallel but have high complexity [3], [8], [9]. In the literature, different methods have been proposed to construct NB-Polar codes, which can be classified into two groups. In the first group, the construction uses a kernel transformation of order 2 [2], [4], [5]. In the case of the second group, the kernel transformation is of order higher than 2 [3], [6], [7], and the encoding and decoding have an increase in complexity compared to the kernel of order 2.

The new generation of communication systems, such as the sixth-generation (6G) system, will need to support short packet traffic [10]–[14]. The association of NB-Polar codes and the Cyclic Code-Shift Keying (CCSK) modulation [15] to transmit short packets is examined in [16], [17]. In [17], the authors also propose the Successive Cancellation Min-Sum (SC-MS) decoder, defined in the Log-Likelihood Ratio (LLR) domain, and which is less complex than the Successive Cancellation (SC) decoder. The complexity of a check node (CN) is $\mathcal{O}(q^2)$ while a variable node (VN) has complexity of $\mathcal{O}(q)$.

In this paper, we consider the association of the CCSK modulation and NB-Polar codes to have good decoding performance at ultra-low signal-to-noise ratios (SNRs), which is relevant for low power networks requiring long range connectivity. In addition, we use the kernel transformation of order 2 defined as $\mathcal{T} : (x_0, x_1) = (u_0 \oplus u_1, h \odot u_1)$ [2], where $(\oplus, \odot)$ denote (addition, multiplication) operations over $\mathbb{F}_q$, and h, called kernel coefficient, is a random element of the set $\mathbb{F}_q \backslash \{0\}$. A method to optimize the kernel coefficients was proposed in [16]. In this paper, we show that in practice, choosing the kernel coefficient equal to $h = 1$ does not degrade the performance of the decoder. Hence, the multiplication '$\odot$' is no longer necessary for the construction, encoding, and decoding of NB-Polar codes. That is, we no longer need a memory to store the kernel coefficients and a circuit that performs the operation '$\odot$' during encoding and decoding. Thus, the choice $h = 1$ reduces the complexity of the encoder and decoder.

To further reduce the complexity of the decoder, we consider the SC-MS decoder. In addition, we select only a small set of most meaningful values at the CN processing to optimize the decoder, this helps us achieve a considerable complexity reduction. For $q = 64$, we observe that the CN can achieve a complexity reduction of around 75% making NB-Polar codes more attractive for several applications such as 5G and 6G standards. Our numerical results show that the optimized SC-MS decoder presents a negligible performance degradation with respect to the SC decoder.

The outline of the paper is as follows. In Section II, we present the NB-Polar codes and the CCSK modulation. In Section III, we describe the association of the CCSK modulation and the NB-Polar codes, and we briefly discuss SC-MS decoders. In Section IV, we show that setting all kernel coefficients equal to 1 does not degrade error-correcting performance. We also present new update rules to reduce the complexity of the SC-MS decoder. Finally, Section V concludes the paper.

II. Background

Let $\mathbb{F}_q$ be the GF of order $q = 2^p$, where $p > 1$ and let $\mathbb{F}_q^*$ be the set of all non-zero elements of $\mathbb{F}_q$. A (N, K) NB-Polar code over $\mathbb{F}_q$ has a length of $N = 2^n$, K information symbols, $N - K$ frozen symbols, and code rate $R_c = K/N$. Let $\boldsymbol{u} = (u_0, ..., u_{N-1})$, $u_i \in \mathbb{F}_q$ for $i = 0, ..., N-1$, denote the uncoded symbols. Also, let $\boldsymbol{x} = (x_0, ..., x_{N-1})$, $x_i \in \mathbb{F}_q$ for $i = 0, ..., N-1$, denote the coded symbols before CCSK modulation. After applying the CCSK technique to $\boldsymbol{x} = (x_0, ..., x_{N-1})$, $\boldsymbol{\eta} = (\eta_{x_0}, ..., \eta_{x_{N-1}})$ is obtained and it is sent through a noisy channel, see section II-B and

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section III-A. The output of the channel is $\boldsymbol{y} = (y_0, ..., y_{N-1})$, $y_i = (y_i(0), ..., y_i(q-1))$ for $i = 0, ..., N-1$.

A. Non-Binary Polar Codes

In this paper, the construction of NB-Polar codes is carried out using the kernel transformation $\mathcal{T} : (x_0, x_1) = (u_0 \oplus u_1, h \odot u_1)$, where the kernel coefficient h is an element of the set $\mathbb{F}_q^*$. In [2], [18], the authors have shown that $\mathcal{T}$ guarantees polarization of the NB-Polar codes if q is a prime power.

To construct a NB-Polar code of length $N = 2^n$, $\mathcal{T}$ is used recursively n times, hence, we obtain a structure composed of n layers of kernel coefficients. A method that optimizes the kernel coefficients at each step of the recursion is proposed in [16]. In Fig. 1, we can see the graph of a NB-Polar code of length $N = 2^3$ composed of 3 layers of kernel coefficients, we can also observe variable nodes (VNs) represented by $\boxed{=}$ and check nodes (CNs) represented by $\bigoplus$. VNs and CNs are used in the decoding process.

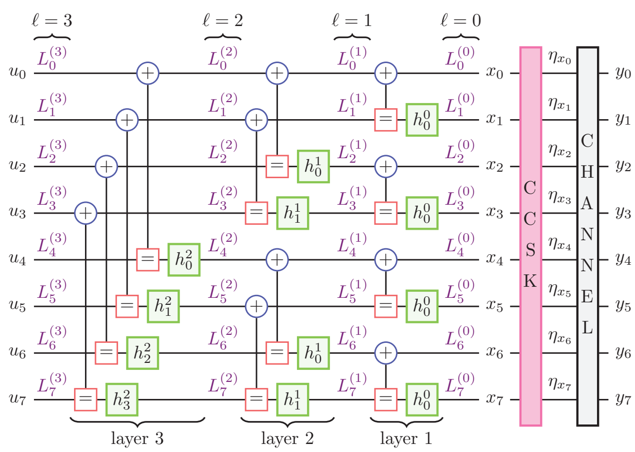


Fig. 1: Graph of a NB-Polar code of length $N = 8$.

For a (N, K) NB-Polar code, the set of indices $\mathcal{I} = \{i_0, ..., i_{K-1}\}$, where $0 \leq i_j \leq N-1$ for $j = 0, ..., K-1$, indicates the positions of the information symbols. In this paper, $\mathcal{I}$ is calculated at each SNR value using a genie-aided successive-cancellation decoder [1], and the frozen symbols are set to zero. Note that the positions of the frozen symbols are known for both encoder and decoder.

B. Cyclic Code-Shift Keying Modulation

The CCSK modulation maps a p-bit symbol to a pseudo-random noise (PN) sequences of length $q = 2^p$, hence, the rate of the modulation is given by $R_m = p/q$. Let $\eta_0 = (\eta_0(0), ..., \eta_0(q-1))$, $\eta_0(k) \in \{-1, +1\}$ for $k = 0, ..., q-1$, be the fundamental PN sequence. The PN sequence $\eta_s = (\eta_s(0), ..., \eta_s(q-1))$ is derived from η_0 performing a circular shift to the left of η_0 in $s \in \{0, ..., q-1\}$ positions:

$$\eta_s(k) = \eta_0((k+s) \mod q) \quad \forall k = 0, ..., q-1.$$

In this paper we used a Linear Feedback Shift Register (LFSR) to construct PN sequences that have good autocorrelation properties. But we can also use a genetic algorithm to generate and optimize a PN sequence.

III. CCSK MODULATION ASSOCIATED TO NB-POLAR CODES

The recursive structure of a NB-Polar code of length $N = 2^n$ allows us to perform the encoding and decoding in $n+1$ stages. Let $\ell \in \{0, ..., n\}$ denote the decoding stage. Let us denote by $\boldsymbol{L}^{(\ell)} = (L_0^{(\ell)}, ..., L_{N-1}^{(\ell)})$ the LLRs computed at stage ℓ during the decoding process, where $L_i^{(\ell)} = (L_i^{(\ell)}(0), ..., L_i^{(\ell)}(q-1))$, $i = 0, ..., N-1$, denotes a vector of LLR. We can see in Fig. 1 the LLRs calculated in each stage of decoding. We compute two types of LLRs, the *channel LLRs* computed in the initialization stage ($\ell = 0$), and the *internal LLRs* calculated in stage $\ell \in \{1, ..., n\}$.

A. Log-Likelihood Ratio

For the CCSK modulation, we consider that $\eta_0 = (\eta_0(0), ..., \eta_0(q-1))$ is a fundamental PN sequence. After the encoding of $\boldsymbol{u}$, each coded symbol $x_i \in \mathbb{F}_q$, $i = 0, ..., N-1$, of $\boldsymbol{x}$ is mapped to the PN sequence $\eta_{x_i} = (\eta_{x_i}(0), ..., \eta_{x_i}(q-1))$ using the expression

$$\eta_{x_i}(k) = \eta_0((k + x_i) \mod q) \quad \forall k = 0, ..., q-1, \tag{1}$$

where $\eta_{x_i}(k) \in \{-1, +1\}$. After the CCSK modulation, the code rate is $R_e = R_c R_m = (Kp)/(Nq)$, and $\boldsymbol{\eta} = (\eta_{x_0}, ..., \eta_{x_{N-1}})$ is transmitted over the Binary Input Additive White Gaussian Noise (BI-AWGN) channel with noise variance σ^2. The channel output $\boldsymbol{y} = (y_0, ..., y_{N-1})$, with $y_i = (y_i(0), ..., y_i(q-1))$ for $i = 0, ..., N-1$, is modeled by $y_i(k) = \eta_{x_i}(k) + z_i(k)$, $k = 0, \ldots, q-1$, where $z_i(k)$ is a sequence of independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance σ^2. We can define the vector of channel LLRs $L_i = (L_i(0), ..., L_i(q-1))$, $i = 0, ..., N-1$, of a GF symbol x_i as

$$L_i = \log\left(\frac{\Pr(y_i \mid \hat{\eta}_{x_i})}{\Pr(y_i \mid \eta_{x_i})}\right), \tag{2}$$

where $\hat{\eta}_{x_i}$ is the sign of y_i, *i.e.* $\hat{\eta}_{x_i}(k) = \text{sign}(y_i(k)) \in \{-1, +1\}$. With $\Pr(y \mid x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x)^2/2\sigma^2}$ for the BI-AWGN channel, we have

$$L_i(x_i) = \sum_{k=0}^{q-1} \frac{y_i(k)}{\sigma^2} \left(\hat{\eta}_{x_i}(k) - \eta_{x_i}(k)\right) \quad \forall x_i \in \mathbb{F}_q. \tag{3}$$

(3) computes only positive LLRs. In the initialization stage, at least one element of L_i should be equal to zero when the decoder operates in the LLR domain, hence, $L_i^{(0)}$ is given by

$$L_i^{(0)}(x_i) = L_i(x_i) - \min(L_i) \quad \forall x_i \in \mathbb{F}_q. \tag{4}$$

B. Successive-Cancellation Min-Sum Decoder

The SC-MS decoder [17], exclusively formulated in the LLR domain, is a simplified version of the SC decoder.

We refer to [17] for a complete presentation of the SC decoder.

For a NB-Polar of length $N = 2^n$, let $\hat{u}_i^{(\ell)}$, $i = 0, ..., N-1$, denote the estimated symbol of $u_i^{(\ell)}$, where $u_i^{(n)} = u_i$ (resp. $u_i^{(0)} = x_i$) is the uncoded (resp. coded) symbol. Let also consider the transformation

$$\mathcal{T}: \left(u^{(\ell-1)}_{\theta_t^{(\ell-1)}}, u^{(\ell-1)}_{\phi_t^{(\ell-1)}}\right) = \left(u^{(\ell)}_{\theta_t^{(\ell-1)}} \oplus u^{(\ell)}_{\phi_t^{(\ell-1)}}, h^{(\ell-1)}_{\phi_t^{(\ell-1)}} \odot u^{(\ell)}_{\phi_t^{(\ell-1)}}\right), \tag{5}$$

where $\theta_t^{(\ell-1)} = 2t - (t \mod 2^{(\ell-1)})$ and $\phi_t^{(\ell-1)} = 2^{(\ell-1)} + 2t - (t \mod 2^{(\ell-1)})$, for $t = 0, 1, ..., N/2-1$.
To simplify the notations in the paper, we use θ, ϕ, and h to denote $\theta_t^{(\ell-1)}$, $\phi_t^{(\ell-1)}$, and $h^{(\ell-1)}_{\phi_t^{(\ell-1)}}$, respectively.
The update rule at a CN is given by

$$L'_\theta\left(u_\phi^{(\ell)}\right) = L_\theta^{(\ell-1)}\left(u_\theta^{(\ell)} \oplus u_\phi^{(\ell)}\right) + L_\phi^{(\ell-1)}\left(h \odot u_\phi^{(\ell)}\right) \; \forall u_\phi^{(\ell)} \in \mathbb{F}_q, \tag{6}$$

$$L_\theta^{(\ell)}(u_\theta^{(\ell)}) = \min\left(L'_\theta\right) \quad \forall u_\theta^{(\ell)} \in \mathbb{F}_q, \tag{7}$$

The update rule at a VN is defined as

$$L'_\phi\left(u_\phi^{(\ell)}\right) = L_\theta^{(\ell-1)}\left(\hat{u}_\theta^{(\ell)} \oplus u_\phi^{(\ell)}\right) + L_\phi^{(\ell-1)}\left(h \odot u_\phi^{(\ell)}\right) \; \forall u_\phi^{(\ell)} \in \mathbb{F}_q, \tag{8}$$

$$L_\phi^{(\ell)}\left(u_\phi^{(\ell)}\right) = L'_\phi\left(u_\phi^{(\ell)}\right) - \min\left(L'_\phi\right) \; \forall u_\phi^{(\ell)} \in \mathbb{F}_q. \tag{9}$$

Equation (9) is used to avoid the very large numerical values of internal LLRs. The estimated symbols are propagated with

$$\left(\hat{u}_\theta^{(\ell-1)}, \hat{u}_\phi^{(\ell-1)}\right) = \left(\hat{u}_\theta^{(\ell)} \oplus \hat{u}_\phi^{(\ell)}, h \odot \hat{u}_\phi^{(\ell)}\right). \tag{10}$$

By applying (6), (7), (8), (9), and (10), we can estimate $\hat{\boldsymbol{u}} = (\hat{u}_0^{(n)}, ..., \hat{u}_{N-1}^{(n)})$ as follows

$$\hat{u}_i^{(n)} = \begin{cases} \arg\min\limits_{u_i^{(n)} \in \mathbb{F}_q} L_i^{(n)}\left(u_i^{(n)}\right), & \text{if } i \in \mathcal{I}, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

It is shown in [17] that the SC-MS decoder is less complex than the SC decoder.

IV. Low Complexity NB-Polar Codes

In this section, we simplify the kernel transformation for construction of the NB-Polar codes. We also reduce the complexity of the SC-MS decoder.

A. Optimized Encoding of NB-Polar Codes

Once the construction of a NB-Polar code of length $N = 2^n$ is finished, we obtain n layers of kernel coefficients and the number of kernel coefficients in each layer is given by $2^{\psi-1}$, where ψ is the layer number. Let d_ω denote the configuration where the first ω layers of h are equal to 1 counting from left to right. Hence, d_0 means that all kernel coefficients are chosen randomly, and d_n means that all kernel coefficients are equal to 1. For the NB-Polar code of $N = 2^3$ shown in Fig. 1, Fig. 2 depicts all possible configurations for d_ω.
To choose among these configurations d_ω, we take into account the association of CCSK modulation and NB-Polar codes. We compare the FER performance results of the SC decoder by varying d_ω, and we choose the configuration d_ω with the largest possible number of coefficients equal to 1 that does not degrade the FER performance of the SC decoder.

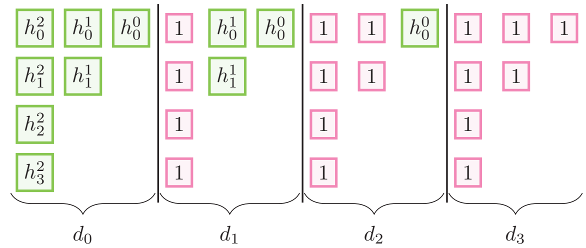


Fig. 2: All possible configurations of d_ω.

In Fig. 3, we present the FER performance results for $d_\omega = d_0$ and $d_\omega = d_n$ considering different code lengths $N = 2^n$, different code rates $R_c = K/N$, and different fields $\mathbb{F}_q$. We can clearly see that the FER performance of the SC decoder is the same for any configuration d_ω. Hence, $d_\omega = d_n$ is the best choice, that is, we set all kernel coefficients to 1. We also performed Monte Carlo simulations for a $(N = 64, K = 20)$ NB-Polar code and for $d_\omega \in \{d_0, d_n\}$ over $\mathbb{F}_q \in \{\mathbb{F}_{2^6}, \mathbb{F}_{2^7}, \mathbb{F}_{2^8}, \mathbb{F}_{2^9}\}$, once again we obtained that $d_\omega = d_n$ does not degrade the FER performance, thus $d_\omega = d_n$ is the best choice.
From all the results obtained, we can conclude that with the CCSK modulation we do not need kernel coefficients $h > 1$. Hence, all kernel coefficients can be equal to 1, this implies that we can simply remove the total memory dedicated to storing the kernel coefficients. Also, in the process of construction, encoding, and decoding, the multiplication '$\odot$' is no longer necessary, and $\mathcal{T}: (x_0, x_1) = (u_0 \oplus u_1, h \odot u_1)$ becomes $\mathcal{T}_O: (x_0, x_1) = (u_0 \oplus u_1, u_1)$. Thus, encoding of NB-Polar codes only needs to use XOR logic gates when $\mathcal{T}_O$ is used as kernel transformation. We can note that the circuit that performs the operation '$\odot$' for encoding/decoding is no longer necessary, and the transformation given in (5) becomes:

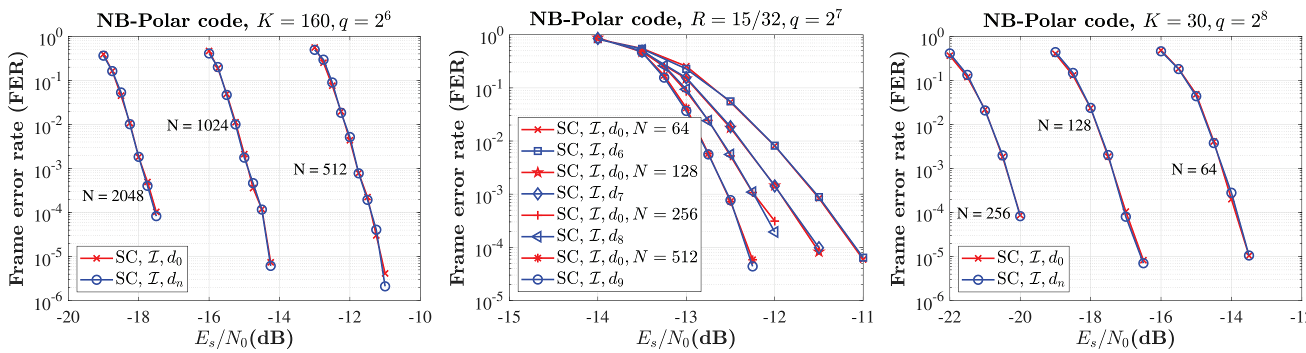
$$\mathcal{T}_O: \left(u^{(\ell-1)}_{\theta_t^{(\ell-1)}}, u^{(\ell-1)}_{\phi_t^{(\ell-1)}}\right) = \left(u^{(\ell)}_{\theta_t^{(\ell-1)}} \oplus u^{(\ell)}_{\phi_t^{(\ell-1)}}, u^{(\ell)}_{\phi_t^{(\ell-1)}}\right). \tag{12}$$

B. Optimized Decoding of NB-Polar Codes

In this section, we present the update rules used to reduce decoding complexity of SC-MS decoders. Let n_o be a natural number such that $0 < n_o \leq q$, and let $\mathbb{F}_{n_o}^{(\ell)} = \{\lambda_0^{(\ell)}, \lambda_1^{(\ell)}, ..., \lambda_{n_o-1}^{(\ell)}\}$ be a subset of $\mathbb{F}_q$, where $\lambda_k^{(\ell)} \in \mathbb{F}_q$ for $k = 0, ..., n_o - 1$. Let $L_{\phi_O}^{(\ell)} = (L_{\phi_O}^{(\ell)}(\lambda_0^{(\ell)}), ..., L_{\phi_O}^{(\ell)}(\lambda_{n_o-1}^{(\ell)}))$ denote the n_o smallest values of the vector $L_\phi^{(\ell)} = (L_\phi^{(\ell)}(0), ..., L_\phi^{(\ell)}(q-1))$. With these notations and considering $\mathcal{T}_O$, the update rule at a CN is given by

$$L'_\theta\left(\lambda_k^{(\ell)}\right) = L_\theta^{(\ell-1)}\left(u_\theta^{(\ell)} \oplus \lambda_k^{(\ell)}\right) + L_{\phi_O}^{(\ell-1)}\left(\lambda_k^{(\ell)}\right) \; \forall \lambda_k^{(\ell)} \in \mathbb{F}_{n_o}^{(\ell)}, \tag{13}$$

$$L_\theta^{(\ell)}(u_\theta^{(\ell)}) = \min\left(L'_\theta\right) \quad \forall u_\theta^{(\ell)} \in \mathbb{F}_q, \tag{14}$$



(a) $K = 160$ and $\mathbb{F}_{2^6}$. (b) $R = 15/32$ and $\mathbb{F}_{2^7}$. (c) $K = 30$ and $\mathbb{F}_{2^8}$.

Fig. 3: FER performance of SC decoders for $d_\omega \in \{d_0, d_n\}$ over $\mathbb{F}_q \in \{\mathbb{F}_{2^6}, \mathbb{F}_{2^7}, \mathbb{F}_{2^8}\}$.

For a fixed value of $u_\theta^{(\ell)}$, (13) and (14) perform $n_o \leq q$ operations, while (6) and (7) perform q operations. Hence the importance of choosing a small value of n_o.

For the case of the VN, the update rule is given by

$$L'_\phi\left(u_\phi^{(\ell)}\right) = L_\theta^{(\ell-1)}\left(\hat{u}_\theta^{(\ell)} \oplus u_\phi^{(\ell)}\right) + L_\phi^{(\ell-1)}\left(u_\phi^{(\ell)}\right) \; \forall u_\phi^{(\ell)} \in \mathbb{F}_q, \tag{15}$$

$$L_\phi^{(\ell)}\left(u_\phi^{(\ell)}\right) = L'_\phi\left(u_\phi^{(\ell)}\right) - \min\left(L'_\phi\right) \; \forall u_\phi^{(\ell)} \in \mathbb{F}_q. \tag{16}$$

We can estimate $\hat{\boldsymbol{u}} = (\hat{u}_0^{(n)}, ..., \hat{u}_{N-1}^{(n)})$ by applying (13), (14), (15), (16), (10), and (11).

The value of n_o is optimized using Monte Carlo simulations. To compare the FER performance of the optimized decoders, the SC decoder performance is shown as a benchmark using $n_o = q$ and $d_\omega = d_0$. We consider a performance loss of less than 0.2 dB to be negligible.

Fig. 4 shows the FER performance of SC-MS decoders for $(N = 64, K = 20)$ and $d_\omega = d_n$ over $\mathbb{F}_{2^6}$, we observe at FER $= 10^{-4}$ a small performance loss of about 0.19 dB for $n_o = 16$ and 0.11 dB for $n_o = 20$, thus n_o between 16 and 20 is a good choice. Of course, we always choose the smallest possible value of n_o.

Simulation results for $(N = 64, K = 20)$ and $\mathbb{F}_q \in \{\mathbb{F}_{2^6}, \mathbb{F}_{2^7}, \mathbb{F}_{2^8}, \mathbb{F}_{2^9}\}$ are provided in Fig. 5. We can see at FER $= 10^{-4}$ that the performance losses of the SC-MS decoders are around 0.11 dB for $(q = 2^6, n_o = 20)$, 0.15 dB for $(q = 2^7, n_o = 30)$, 0.13 dB for $(q = 2^8, n_o = 40)$, and 0.08 dB for $(q = 2^9, n_o = 50)$. From Fig. 4 and Fig. 5, we can note that for a fixed code rate, the ratio n_o/q decreases as q increases.

Fig. 6 depicts the FER performance of the SC-MS decoders for $N \in \{64, 128, 256\}$ and $R = 5/16$ over $\mathbb{F}_{2^6}$. Comparing the decoders, the SC-MS decoders with small values of n_o can reach the FER performance of the SC decoders, we get at FER $= 10^{-4}$ a small performance loss of around 0.15 dB for $(N = 64, n_o = 18)$, 0.19 dB for $(N = 128, n_o = 24)$, and 0.08 dB for $(N = 256, n_o = 28)$. From these results, we can see that for a fixed code rate and fixed q, n_o needs to increase as the length N increases in order to have negligible loss.

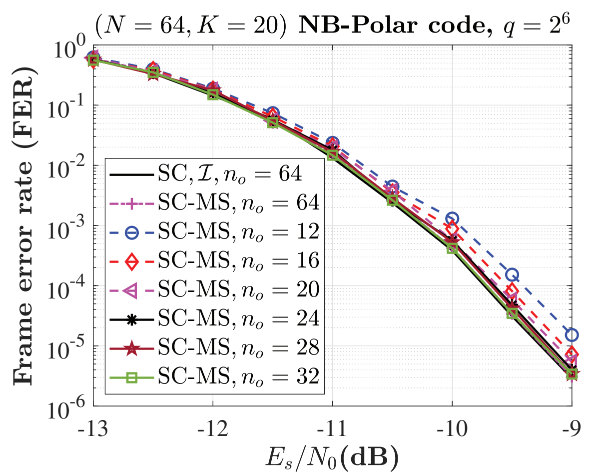


Fig. 4: FER performance of SC-MS decoders for $d_\omega = d_n$, $n_o \in \{12, 16, 20, 24, 28, 32, 64\}$, and $\mathbb{F}_{2^6}$.

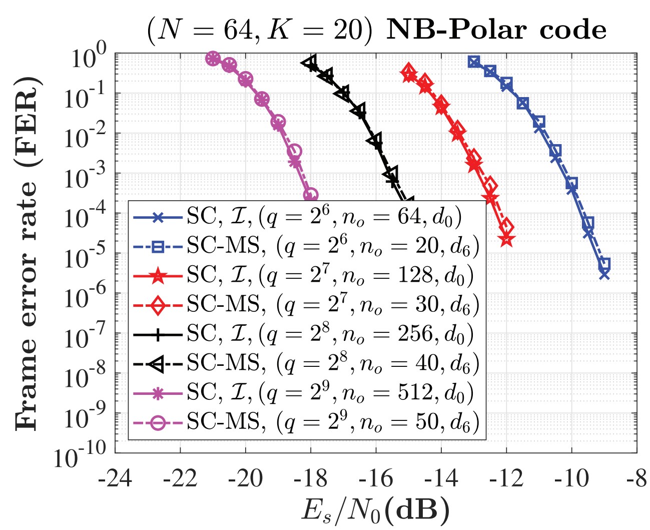


Fig. 5: FER performance of SC-MS decoders for $d_\omega = d_n$, $N = 64$ and $\mathbb{F}_q \in \{\mathbb{F}_{2^6}, \mathbb{F}_{2^7}, \mathbb{F}_{2^8}, \mathbb{F}_{2^9}\}$.

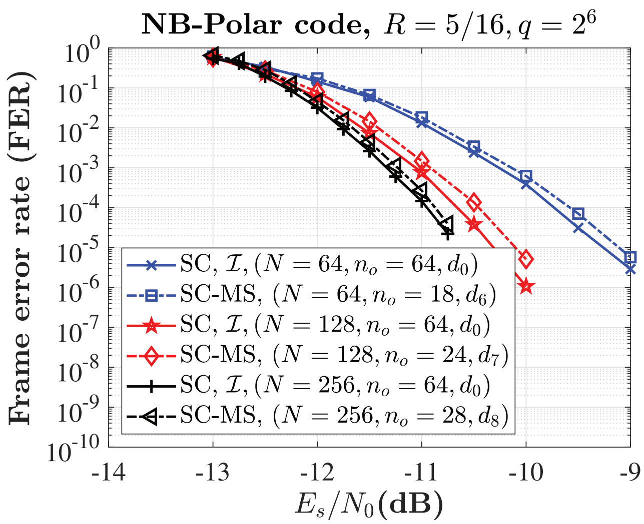


Fig. 6: FER performance of SC-MS decoders for $d_\omega = d_n$, $N \in \{64, 128, 256\}$ and $\mathbb{F}_{2^6}$.

C. Complexity of SC-based Decoders

We can roughly estimate the complexity of the SC-MS decoder and its optimized version by comparing the number of operations required during encoding and decoding. We report in Table I the complexity of a CN and a VN. Note that the subtractions are considered as additions in terms of complexity. We can see that in the SC-MS decoder, a CN requires q^2 additions and $q(q-1)$ comparisons; and a VN requires q additions, $q-1$ comparisons, and q subtractions. When the optimized SC-MS decoder is considered, the complexity of the CN is reduced, we can observe that the number of additions is reduced from q^2 to qn_o and the number of comparisons is reduced from $q(q-1)$ to $q(n_o-1)$.
For the case of the $(N = 64, K = 20)$ NB-Polar code with $q = 64$ and $n_o = 16$, a large reduction in operations can be observed at a CN: around 75% of adders, and 76.2% of comparators.

TABLE I: Complexity of a single CN and a single VN for $\mathbb{F}_q$.

	Node	# of adders	# of comparators
SC-MS	CN	q^2	$q(q-1)$
	VN	$2q$	$q-1$
Optimized SC-MS	CN	qn_o	$q(n_o-1)$
	VN	$2q$	$q-1$

V. Conclusion

In this paper, we have associated the CCSK modulation to NB-Polar codes to achieve very good FER performance at ultra-low SNRs. We have demonstrated that choosing the kernel coefficients equal to 1 does not degrade error-correcting performance and reduces the complexity of the encoder and decoder. To hugely reduce the complexity of the SC-MS decoder, we have used a limited number n_o of meaningful LLR values at the check node update to optimize the decoder. The Monte Carlo simulations have shown that the optimized SC-MS decoder presents a negligible performance degradation with respect to the SC decoder.

Appendix

Considering that $\boldsymbol{x}$ is mapped by the Binary Phase-Shift Keying (BPSK) modulation and transmitted over the BI-AWGN channel. For the NB-Polar code of $N = 2^n$, we find empirically that the optimal value of d_ω is obtained when ω is in the neighborhood of $\lfloor n/2 \rfloor$. Our results show that the kernel coefficients closest to the channel must be different from 1. Simulation results for $N = 2^{10}$, $\mathbb{F}_{2^7}$, and $K \in \{341, 512, 683\}$ are depicted in Fig. 7. We obtain that $d_\omega = d_5$ is a good choice and does not degrade the decoding performance. Note that only 31 kernel coefficients (d_5) are needed instead of 1023 kernel coefficients (d_0), obtaining a memory reduction of 96.97%.

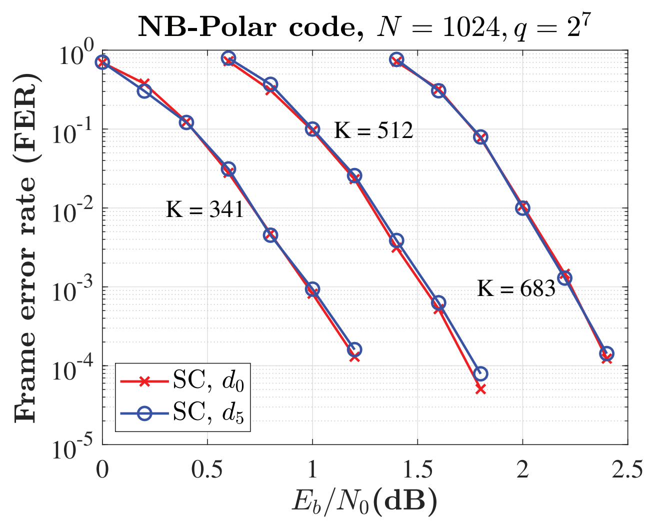


Fig. 7: BPSK modulation + BI-AWGN channel.

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